Randomly driven Korteweg–de Vries–Burgers equation

Arkadiusz Orłowski*
Arbeitsgruppe "Nichtklassische Strahlung" der Max-Planck-Gesellschaft an der Humboldt-Universität zu Berlin, Rudower Chaussee 5, 12489 Berlin, Germany
(Received 3 November 1993)

An exact solution of the randomly driven Korteweg–de Vries–Burgers equation is found and its basic statistical properties are investigated. An approximate formula for the large-time-scale behavior of the mean solution is evaluated.

PACS number(s): 05.40.+j, 52.35.Mw, 52.35.Sb, 52.35.Tc

There is still an increasing amount of attention in the propagation of nonlinear waves in randomly excited media. The interest is continuously stimulated by the remarkable variety of applications in acoustics, hydrodynamics, plasma physics as well as solid-state physics [1]. Unfortunately, analytical results are available only for the simplest models. In the case of the Korteweg–de Vries equation with the external time-dependent white noise

$$U_t + UU_x + \beta U_{xxx} = f(t) \ , \tag{1}$$

the deformation of the one-soliton solution called the diffusion of soliton was found by Wadati [2]. Also behavior of multisoliton solutions in the presence of such a time-dependent excitation was studied [3]. Similar investigations were performed for the stochastic Burgers equation [4–7]

$$U_t + UU_x - \mu U_{xx} = f(t) \ , \tag{2}$$

and a damping of the shock-wave solution was observed. More general Gaussian excitations as, e.g., colored noise were also considered [6,7].

In both cases we have the completely integrable systems [8] disturbed by the spatially-independent random external noise. In this paper I generalize the above-mentioned results to the case of a nonintegrable system investigating the following stochastic KdV-Burgers equation:

$$U_t + UU_x + \beta U_{xxx} - \mu U_{xx} = f(t) \ . \tag{3}$$

The model describes randomly driven nonlinear waves in both dispersive and dissipative media and is a natural extension of the previous cases. In spite of its simplicity it seems to be a good approximation of various multiple-components nonlinear systems (see [9] and references therein). The KdV-Burgers equation does not satisfy the Painlevé property [8] and does not possess any nontrivial prolongation structure [10]. It can be solved neither by the inverse scattering method [11] nor by the Hopf-Cole transformation [12]. Nevertheless, some analytical treatment of the influence of random force on the propagation of nonlinear waves described by this equation is still possible. In the following, I find a formal random solution corresponding to a stationary solution of the unperturbed KdV-Burgers equation and discuss its basic statistical properties.

An essential point is the possibility of reduction of the above equation to the standard force-free KdV-Burgers equation. This remarkable property is caused by the fact that the noise $f(t)$ does not depend on the space coordinate. In the general case the noise can be both time and space dependent but the corresponding stochastic KdV-Burgers equation is too complicated to be investigated analytically, e.g., the possibility of a chaotic behavior should be taken into account. Straightforward calculations show that the following transformations (cf. [2,7] and references therein)

$$V(z,t) = U(x,t) - W(t) \ , \tag{4a}$$

$$W(t) = \int_0^t f(t')dt' \ , \tag{4b}$$

$$z = x - \psi(t) \ , \tag{4c}$$

$$\psi(t) = \int_0^t W(t')dt' \ , \tag{4d}$$

lead to the homogeneous KdV-Burgers equation for $V(z,t)$

$$V_t + VV_z + \beta V_{zzz} - \mu V_{zz} = 0 \ . \tag{5}$$

We assume that the noise $f(t)$ is a Gaussian stochastic process with zero mean $\langle f(t)\rangle = 0$ and a given correlation function

$$\langle f(t)f(t')\rangle = K_f(t,t') \ . \tag{6}$$

Under these assumptions, stochastic integrals in (4) are well definite. In this case, also $W(t)$ and $\psi(t)$ are Gaussian stochastic processes with zero means. We also have

$$\sigma_\psi^2(t) = \langle \psi^2(t)\rangle = \int_0^t \int_0^t \langle W(s)W(s')\rangle ds\, ds', \tag{7}$$

where

*Permanent address: Instytut Fizyki, Polska Akademia Nauk, Aleja Lotników 32/46, 02-668 Warszawa, Poland.

 ©1994 The American Physical Society

$$\langle W(s)W(s')\rangle = \int_0^s \int_0^{s'} K_f(u,u')du\,du'. \quad (8)$$

It is known that the homogeneous KdV-Burgers equation must possess a stationary solution (see, e.g., [13,14]). I use in this paper an explicit form of such a solution found rather recently [15] by means of the interesting combination of the Hirota method [16] and the singular manifold expansion [17]. It can also be obtained from the stationary solution of the Fischer equation [18]. Changing denotations, we rewrite the solution as follows:

$$V(z,t) = \frac{6}{5}\mu k + 3\beta k^2 \operatorname{sech}^2\frac{k}{2}\left(z - \frac{6}{5}\mu k t\right) - \frac{6}{5}\mu k \tanh\frac{k}{2}\left(z - \frac{6}{5}\mu k t\right) , \quad (9)$$

where $k = \pm\frac{\mu}{5\beta}$. Let us note that the above solution is an interesting *linear* superposition of a "solitary wave" which resembles the one-soliton solution of the KdV equation and a "shock wave" resembling the stationary solution of the Burgers equation. Although it is only a particular solution corresponding to a special initial condition, it enables us to see in more detail what happens during the randomly driven evolution.

Making use of transformations (4) we immediately derive an exact expression for the random field obeying our stochastic KdV-Burgers equation

$$U(x,t) = W(t) + \frac{6}{5}\mu k + 3\beta k^2 \operatorname{sech}^2\frac{k}{2}\xi - \frac{6}{5}\mu k \tanh\frac{k}{2}\xi , \quad (10)$$

where $\xi = x - \frac{6}{5}\mu k t - \psi(t)$ is sometimes called the "fluctuating coordinate" [3].

The above equation depends on the time-varying random parameter determined by the external noise and we should perform the averaging with respect to different realizations of the noise to get physically interpretable results. It is seen that the randomized solution (10) forms the *nonlinear* transformation of the noise. Fortunately, due to the Gaussian character of the noise $f(t)$, we can find the mean solution via direct integration of this equation with respect to the appropriate Gaussian probabilistic measure

$$\langle U(x,t)\rangle = \langle W(t)\rangle + \frac{1}{\sqrt{2\pi\sigma_\psi^2(t)}}\int_{-\infty}^{+\infty} V(x-\psi,t) \times \exp\left(-\frac{\psi^2}{2\sigma_\psi^2(t)}\right) d\psi . \quad (11)$$

Let us assume for simplicity that the random external force $f(t)$ is the Gaussian white noise

$$K_f(t,t') = 2D\delta(t'-t), \qquad D > 0 . \quad (12)$$

In this case we have

$$\sigma_\psi^2(t) = \langle m^2(t)\rangle = \tfrac{2}{3}Dt^3 . \quad (13)$$

Using the fact that $\langle W(t)\rangle = 0$ and some approximations developed in [6,7] we obtain the following large-time-scale behavior of the mean solution:

$$\langle U(x,t)\rangle = \frac{6\sqrt{3}}{\sqrt{\pi D t^3}}\exp(-z^2) + \frac{6\mu k}{5}[1 - \operatorname{erf}(z)] , \quad (14)$$

where

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}}\int_0^z \exp(-x^2)dx , \quad (15)$$

is the well known error function [19] and

$$z = \frac{\sqrt{3}}{2\sqrt{Dt^3}}(x - \tfrac{6}{5}\mu k t) . \quad (16)$$

For sufficiently large time, this expression correctly describes the deformation of the stationary solution of the KdV-Burgers equation during its propagation in randomly excited media. The observed behavior is a superposition of the effect called the diffusion of soliton and a kind of damping of the shock wave which was first correctly described in [5] and independently in [6,7].

In summary, propagation of the stationary solution of the KdV-Burgers equation in randomly excited media was studied. Although the model equation is not completely integrable it was possible to find the analytical solution. It can be used for the verification of various approximate techniques. The approximate formula describing large-time-scale behavior of the averaged solution is also evaluated. The analytical treatment was possible due to the existence of transformations which replace the forced equation by the standard one in new coordinates. These transformations have a long and interesting history [20] and some related problems will be studied elsewhere.

I thank D. J. Kaup and A. Kujawski for interesting comments. I am also grateful to K. Sobczyk for some earlier discussions.

[1] F. G. Bass, Yu. S. Kivshar, V. V. Konotop, and Yu. A. Sinitsyn, Phys. Rep. **157**, 65 (1988).
[2] M. Wadati, J. Phys. Soc. Jpn. **52**, 2642 (1983).
[3] M. Wadati and Y. Akutsu, J. Phys. Soc. Jpn. **53**, 3342 (1984).
[4] H. Konno, J. Phys. Soc. Jpn. **54**, 4475 (1985).
[5] M. Błaszak, Acta Phys. Pol. A **70**, 523 (1986).
[6] A. Orłowski, Master thesis, Warsaw Technological University, 1987 (unpublished).
[7] A. Orłowski and K. Sobczyk, Rep. Math. Phys. **27**, 59 (1989).
[8] M. J. Ablowitz and P. A. Clarkson, *Solitons, Nonlinear*

Evolution Equations and Inverse Scattering (Cambridge University Press, Cambridge, 1991).
[9] H. Konno and H. Soneda, J. Phys. Soc. Jpn. **57**, 1163 (1988).
[10] F. Feudel and H. Steudel, Phys. Lett. **107A**, 5 (1985).
[11] M. J. Ablowitz and H. Segur, *Solitons and the Inverse Scattering Transform* (SIAM, Philadelphia, 1981).
[12] E. Hopf, Comm. Pure Appl. Math. **3**, 201 (1950); J. D. Cole, Q. Appl. Math. **9**, 225 (1951).
[13] A. Kitada and H. Umehara, J. Phys. Soc. Jpn. **57**, 1855 (1988).
[14] V. I. Karpman, *Nonlinear Waves in Dispersive Media* (Pergamon, Oxford, 1976).
[15] K. Nozaki, J. Phys. Soc. Jpn. **56**, 3052 (1987).
[16] R. Hirota, Phys. Rev. Lett. **27**, 1192 (1971).
[17] J. Weiss, M. Tabor, and G. Carnevale, J. Math. Phys. **24**, 522 (1983).
[18] M. J. Ablowitz and A. Zeppetella, Bull. Math. Biol. **41**, 835 (1979).
[19] *Handbook of Mathematical Functions*, edited by M. Abramowitz and I. A. Stegun (Dover, New York, 1972).
[20] R. M. Miura, J. Math. Phys. **9**, 1202 (1968).